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LETTER TO THE EDITOR

# On one approach to an electromagnetic diffraction problem in a wedge shaped region

M A Lyalinov

Department of Mathematical Physics, Institute of Physics, University of St Petersburg, 198904, Russia

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**Abstract.** An electromagnetic diffraction problem in a wedge shaped region is reduced to a system of coupled functional equations by means of Sommerfeld integrals. Anisotropic impedance boundary conditions are satisfied on the wedge's faces. This system of functional equations is solved by regular perturbation method. It is shown that for weak anisotropy the solution is presented by converging series which are Neumann recurrent series for linear equations with contracting operators. In a general case, the problem is reduced to linear equations with compact operators. The wave field asymptotic is computed for the region outside the vicinity of the edge of the wedge.

The problem of diffraction by a perfectly conducting wedge was solved in classical works of Sommerfeld. Maliuzhinets (1959) studied diffraction by imperfectly conducting wedges and proposed regular solving procedure via reduction to functional equations which are equivalent to scalar Riemann problem for analytic functions (Senior 1959, Williams 1959). Recently specific interest to diffraction by coated wedges and smooth surfaces has arisen due to applications of such models in radiophysics and acoustics (Tuzhilin 1973, Bernard 1987, Senior 1992, Buldyrev and Ljalinov 1992). We study the problem of diffraction by a wedge with anisotropic face impedances (figure 1). It cannot be solved exactly in the general case, since it is equivalent to matrix Riemann problem for analytic vectors. However, we develop an effective approach to the problem which can also be applied in different problems in wedge shaped regions. Therefore, we study this problem as a simple non-trivial example of applying the proposed approach which has more general meaning for the investigation of the wave field in wedge shaped regions.

Let harmonic electromagnetic plane wave with $e^{-i\omega t}$ time dependence, which is suppressed in this letter,

$$E_z^i = E_g \exp(-ik\rho\cos(\phi-\phi_0)) \qquad H_z^i = E_f \exp(-ik\rho\cos(\phi-\phi_0)) \tag{1}$$

be incident on a wedge with anisotropic impedance boundary conditions (Kurushin *et al* 1975)

$$\frac{i}{\omega\epsilon_0}\left(\pm\frac{1}{\rho}\frac{\partial H_z}{\partial\varphi}\right)_{\pm\phi} = \sqrt{\frac{\mu_0}{\epsilon_0}}\left\{a_{11}^{\pm}\frac{i}{\omega\mu}\left(\pm\frac{1}{\rho}\frac{\partial E_z}{\partial\varphi}\right) - a_{12}^{\pm}H_z\right\}_{\pm\phi}$$

$$(E_z)_{\pm\phi} = -\sqrt{\frac{\mu_0}{\epsilon_0}}\left\{a_{21}^{\pm}\frac{i}{\omega\mu_0}\left(\pm\frac{1}{\rho}\frac{\partial E_z}{\partial\varphi}\right) - a_{22}^{\pm}H_z\right\}_{\pm\phi} \tag{2}$$

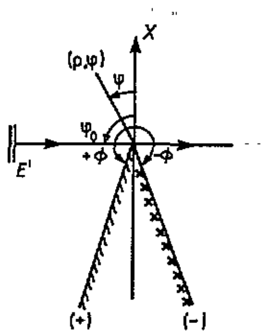


**Figure 1.** Diffraction of a plane wave by a wedge.

where cylindrical coordinates $(\rho, \varphi, z)$ are introduced, and $z$ axis directed along the edge (figure 1). The signs $(\pm)$ in (2) correspond to $\varphi = \pm\phi$, respectively, and matrix $A = \{a_{ik}\}_{i,k=1}^{2}$ is the matrix of surface impedances which are computed via dielectric and magnetic constants of the anisotropic coatings (Kurushin *et al* 1975). The components $E_z$, $H_z$ are independent on $z$ coordinate and can be determined as the solutions of stationary wave equations.

$$\Delta E_z + k^2 E_z = 0 \qquad \Delta H_z + k^2 H_z = 0 \qquad k = \sqrt{\epsilon_0 \mu_0}\omega. \tag{3}$$

The other components of electromagnetic field are expressed explicitly from Maxwell equations, if $E_z$ and $H_z$ are known. We seek solutions of wave equations (3) in the form of Sommerfeld integrals

$$\begin{aligned} E_z(\rho, \varphi) &= \frac{1}{2\pi i} \int_\gamma g(\alpha + \varphi) \exp(-\mathrm{i}k\rho \cos\alpha)\, \mathrm{d}\alpha \\ H_z(\rho, \varphi) &= \frac{1}{2\pi i} \int_\gamma f(\alpha + \varphi) \exp(-\mathrm{i}k\rho \cos\alpha)\, \mathrm{d}\alpha \end{aligned} \tag{4}$$

where $\gamma$ is well known Sommerfeld double loop contour (figure 2). We have to add Meixner condition at the edge $(\rho \to 0)$ and special condition at infinity which is more convenient formulated for spectral functions $f$, $g$. The representations (4), satisfying wave equations (4), are substituted into the boundary conditions (2), and, using Maliuzhinets theorem, we obtain

$$\begin{aligned} &\pm \sin\alpha(f(\alpha \pm \phi) + f(-\alpha \mp \phi)) + a_{12}^{\pm}(f(\alpha \pm \phi) - f(-\alpha \pm \phi)) \\ &\qquad = \pm \lambda a_{11}^{\pm} \sqrt{\epsilon_0/\mu_0} \sin\alpha(g(\alpha \pm \phi) + g(-\alpha \pm \phi)) \\ &\pm a_{21}^{\pm} \sin\alpha(g(\alpha \pm \phi) + g(-\alpha \mp \phi)) + (g(\alpha \pm \phi) - g(-\alpha \pm \phi)) \\ &\qquad = \lambda a_{22}^{\pm} \sqrt{\mu_0/\epsilon_0}(f(\alpha \mp \phi) - f(-\alpha \pm \phi)) \end{aligned} \tag{5}$$

where we have voluntarily introduced the parameter $\lambda(|\lambda| \leqslant 1)$, which must be equal to unit in resulting formulae. The meaning of the introduced artificial parameter is to develop perturbation methods to treat the functional equations (5). The system (5) has to be supplemented by conditions for spectral functions which are as follows:

- $f(\alpha) - E_f/(\alpha - \varphi_0), g(\alpha) - E_g/(\alpha - \varphi_0)$ are regular and bounded in the strip $\bar{\Pi} = \{\alpha \in C : |\mathrm{Re}(\alpha)| \leqslant \phi\}$.
- Spectral function $f(\alpha)$ satisfies condition

$$|f(\alpha) - f(\pm i\infty)| < \exp(-\delta|\mathrm{Im}(\alpha)|) \qquad 0 < \delta < 1 \tag{6}$$

when $\alpha \in \bar{\Pi}, |\mathrm{Im}\,\alpha| \to \infty$. For the function $g(\alpha)$ we have the same inequality.

- $f(-i\infty) = -f(i\infty), g(-i\infty) = -g(i\infty)$.

It can be shown that $f(-i\infty) = iH_z(0, \varphi)/2$, $g(-i\infty) = iE_z(0, \varphi)/2$. So $f(\alpha)$, $g(\alpha)$ are regular and bounded in $\bar{\Pi}$ with fixed residues $E_f$, $E_g$ in the simple pole $\varphi = \varphi_0$ to reproduce the incident plane wave $E_z^i$, $H_z^i$. The initial problem is reduced to the problem (5), (6) for regular functions. Using conformal mapping, one can show that the system of functional equations is transformed to Riemann matrix problem with discontinuous coefficient. However, we shall use another approach to solving (5), which, of course, is equivalent to solving the Riemann problem for analytic functions.

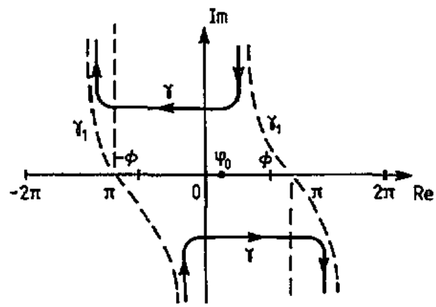


**Figure 2.** Integration contour $\gamma$ and steepest descent paths $\gamma_1$.

We introduce new unknown functions $\xi(\alpha)$, $\zeta(\alpha)$ by equalities

$$f(\alpha) = \psi_f(\alpha)\sigma(\alpha)\xi(\alpha) \qquad g(\alpha) = \psi_g(\alpha)\sigma(\alpha)\zeta(\alpha) \tag{7}$$

where $\sigma(\alpha) = \pi/(2\Phi)\cos(\pi\varphi_0/2\Phi)/[\sin(\pi\alpha/2\Phi) - \sin(\pi\varphi_0/2\Phi)]$ is a meromorphic function with a single pole $\varphi = \varphi_0$ in $\bar{\Pi}$, $\psi_f(\alpha)$, $\psi_g(\alpha)$ are regular in $\bar{\Pi}$ and meromorphic in $C$ functions determined by

$$\psi_f(\alpha) = \psi_\Phi(\alpha + \Phi + \pi/2 - \theta_+)\psi_\Phi(\alpha - \Phi - \pi/2 + \theta_-) \\ \times \psi_\Phi(\alpha + \Phi - \pi/2 + \theta_+)\psi_\Phi(\alpha - \Phi + \pi/2 - \theta_-)$$

and $\psi_g(\alpha)$ is the same as $\psi_f$ with changing $\theta$ by $\chi$. Here, we have introduced notations $\sin\theta_\pm = a_{12}^\pm$, $\sin\chi_\pm = 1/a_{21}^\pm$ with $\mathrm{Im}(\theta_\pm) > 0$, $\mathrm{Im}(\chi_\pm) > 0$, $\mathrm{Re}(\theta_\pm) > 0$, $\mathrm{Re}(\chi_\pm) > 0$ and $\psi_\Phi$ is Maliuzhinets function, defined as the solution of functional equation (Bernard 1987)

$$\psi_\Phi(\alpha + 2\Phi)/\psi_\Phi(\alpha - 2\Phi) = \cot(\alpha/2 + \pi/4).$$

We seek for the regular functions $\xi$, $\zeta$ in the form of series

$$\xi = \sum_{m=0}^{\infty} \lambda^m \xi_m(\alpha) \qquad \zeta = \sum_{m=0}^{\infty} \lambda^m \zeta_m(\alpha). \tag{8}$$

Substituting (7), (8) into (6) and equating the terms of the same power of $\lambda$, we have the following recurrent system

$$\begin{aligned}
&\xi_m(\alpha \pm \Phi) - \xi_m(-\alpha \pm \Phi) = k_1^{\pm}(\alpha)\left[\frac{\zeta_{m-1}(\alpha \pm \Phi)}{(\sin\alpha \pm \sin\chi_\pm)} + \frac{\zeta_{m-1}(-\alpha \pm \Phi)}{(-\sin\alpha \pm \sin\chi_\pm)}\right] \\
&\zeta_m(\alpha \pm \Phi) - \zeta_m(-\alpha \pm \Phi) = k_2^{\pm}(\alpha)\left[\frac{\xi_{m-1}(\alpha \pm \Phi)}{(\sin\alpha \pm \sin\theta_\pm)} - \frac{\xi_{m-1}(-\alpha \pm \Phi)}{(-\sin\alpha \pm \sin\theta_\pm)}\right] \\
&\xi_0 = E_f/\psi_f(\varphi_0) \qquad \zeta_0 = E_g/\psi_g(\varphi_0) \qquad m \geqslant 1
\end{aligned} \tag{9}$$

where

$$k_1^{\pm}(\alpha) = a_{11}^{\pm} \sin\alpha h_\pm(\alpha) \qquad k_2^{\pm}(\alpha) = \pm a_{22}^{\pm} \sin\chi_\pm / h_\pm(\alpha)$$

$$h_\pm(\alpha) = \frac{\psi_g(\alpha \pm \Phi)(\sin\alpha \pm \sin\chi_\pm)}{\psi_f(\alpha \pm \Phi)(\sin\alpha \pm \sin\theta_\pm)}\sqrt{\frac{\varphi_0}{\mu_0}}.$$

Due to the new unknowns $\xi(\alpha)$, $\zeta(\alpha)$ we have obtained recurrent functional equations (9) with constant coefficients. It can be shown that $\xi_m(\alpha \pm \Phi) - \xi_m(-\alpha \pm \Phi) = O(1/\sin\alpha)$ and $\zeta_m(\alpha \pm \Phi) - \zeta_m(-\alpha \pm \Phi) = O(1/\sin\alpha)$, $|\mathrm{Im}(\alpha)| \to \infty$. Applying modified Fourier transformation (Tuzhilin 1973, Bernard 1987) with integration along the imaginary axis, one can write

$$\xi_m(\alpha) = (K_1\zeta_{m-1})(\alpha) \qquad \zeta_m(\alpha) = (K_2\xi_{m-1})(\alpha) \qquad m \geqslant 1 \tag{10}$$

where

$$\begin{aligned}
(K_1 t)(z) = \frac{i}{8\Phi}\int_{iR} \mathrm{d}\alpha \Bigg\{ & k_1^{+}(\alpha)(\sigma_+(\alpha, z) - \sigma_+(\alpha, \varphi_0))\left[\frac{t(\alpha + \Phi)}{(\sin\alpha + \sin\chi_+)} + \frac{t(-\alpha + \Phi)}{(-\sin\alpha + \sin\chi_+)}\right] \\
& - k_1^{-}(\alpha)(\sigma_-(\alpha, z) - \sigma_-(\alpha, \varphi_0))\left[\frac{t(\alpha - \Phi)}{(\sin\alpha - \sin\chi_-)} + \frac{t(-\alpha - \Phi)}{(-\sin\alpha - \sin\chi_-)}\right]\Bigg\} \\
(K_2 t)(z) = \frac{i}{8\Phi}\int_{iR} \mathrm{d}\alpha \Bigg\{ & k_2^{+}(\alpha)(\sigma_+(\alpha, z) - \sigma_+(\alpha, \varphi_0))\left[\frac{t(\alpha + \Phi)}{(\sin\alpha + \sin\theta_+)} - \frac{t(-\alpha + \Phi)}{(-\sin\alpha + \sin\theta_+)}\right] \\
& - k_2^{-}(\alpha)(\sigma_-(\alpha, z) - \sigma_-(\alpha, \varphi_0))\left[\frac{t(\alpha - \Phi)}{(\sin\alpha - \sin\theta_-)} - \frac{t(-\alpha - \Phi)}{(-\sin\alpha - \sin\theta_-)}\right]\Bigg\}
\end{aligned} \tag{11}$$

where $\sigma \pm (\alpha, z) = \sin(\pi\alpha/2\Phi)/[\cos(\pi\alpha/2\Phi) - \sin(\pi z/2\Phi)]$. It is useful to mention that we have obtained the regular functions $\xi_m(\alpha)$, $\zeta_m(\alpha)$ in $\bar{\Pi}$, satisfying the conditions $\xi_m(\varphi_0) = 0$, $\zeta_m(\varphi_0) = 0$, $m \geqslant 1$. This property guarantees that the residues of $f(\alpha)$ and $g(\alpha)$ in the pole $\varphi = \varphi_0$ produce accurate values of $E_z^i$ and $H_z^i$.

Now we are ready to investigate convergence of the series (8). Let us consider Banach space $\tilde{A}(\Pi')$ that is the linear space of regular functions bounded in the strip

$\Pi' = \{\alpha \epsilon C, |\mathrm{Re}\alpha| < \Phi + \delta\}$ for any small $\delta > 0$ with the norm $\|\xi\| = \sup_{\alpha\epsilon\Pi'} |\xi(\alpha)|$. The operators $K_{1,2}$ are the linear bounded operators in $A(\Pi')$, $\|K_{1,2}\| \leqslant C_{1,2}$ and

$$\|\xi_m\| \leqslant C_1 \|\zeta_{m-1}\| \qquad \|\zeta_m\| \leqslant C_2 \|\xi_{m-1}\| \qquad m \geqslant 1 \tag{12}$$

where the estimations (12) can be easily obtained from (10), (11). The constants $C_1$, $C_2$ are computed explicitly by means of the equalities (11). For the sake of compactness we omit the formulae for $C_1$, $C_2$. We ought to mention only that $C_1$, $C_2$ are proportional to the value $\epsilon \sim |a_{11}^{\pm}|$ or $\epsilon \sim |a_{22}^{\pm}|$ respectively as it follows from (11). Therefore, for a weak anisotropy, when $a_{11}$ or $a_{22}$ are small and $\epsilon \ll 1$, we have $C_1 = O(\epsilon)$ or $C_2 = O(\epsilon)$. Inequalities (12) lead to estimations

$$\|\zeta_m\| \leqslant \mathrm{const}(C_1C_2)^m \qquad \|\xi_m\| \leqslant \mathrm{const}(C_1C_2)^m \qquad m \geqslant 1$$

that give a sufficient condition of uniform convergence in the strip $\Pi'$ for the series (8)

$$\lambda C_1 C_2 < 1. \tag{13}$$

The inequality (13) is valid for any anisotropy and small $\lambda$. When $\lambda \to 1 - 0$ the condition (13) leads to

$$C_1 C_2 < 1. \tag{14}$$

The sufficient condition (14) can be guaranteed, for example, when $C_2 = O(\epsilon)$, $\epsilon \ll 1$, that is for weak anisotropy of material coating of the wedge. So, the series (8) are converging for weak anisotropy, but as is hoped the convergence holds in a more general case. Using convergence of the series (8) and taking into account (10) and (14), we get the system of linear equations in Banach space

$$\xi = K_1\zeta + \xi_0 \qquad \zeta = K_2\xi + \zeta_0 \tag{15}$$

where $K_1$, $K_2$ are bounded operators. From the equations (15) one can get

$$\begin{aligned} \xi &= K_1K_2\xi + K_1\zeta_0 + \xi_0 \\ \zeta &= K_2K_1\zeta + K_2\xi_0 + \zeta_0. \end{aligned} \tag{16}$$

From the condition (14) it follows that $K_1K_2$ and $K_2K_1$ are contracting operators with $\|K_1K_2\| < 1$. In other words, the converging series (8) ($\lambda = 1 - 0$) are Neumann series for linear equations with contracting operators in $\tilde{A}(\Pi')$. It can be shown that equations (15), (16) are deduced independently upon perturbation procedure. When $\|K_1K_2\| \geqslant 1$, Neumann procedure fails. In this case, one can reduce (5) to linear equations with bounded operator and prove that it is a compact operator (Kolmogorov and Fomin 1981). Therefore, we can obtain linear equations of the second kind with a compact operator. Fredholm theory can be applied, and one can prove the existence of the solution, using its uniqueness, for any natural parameters of the problem.

In the case of weak anisotropy ($\epsilon \ll 1$), we are restricted by the leading terms of Neumann series. For simplicity we consider $E_z^i$ polarized incident wave, so that $H_z^i = 0$

and $E_f = 0$. Then, $f(\alpha) = \psi_f(\alpha)\sigma(\alpha)\xi_1(\alpha)(1+O(\epsilon^2))$, $g(\alpha) = \psi_g(\alpha)\sigma(\alpha)\zeta_0(1+O(\epsilon^2))$, where $\xi_1(\alpha) = K_1\zeta_0 = O(\epsilon)$ and $\zeta_0 = E_g/\psi_g(\varphi_0)$. For the leading terms we have

$$
\begin{aligned}
E_z(\rho,\varphi) &= \frac{1}{2\pi i}\int_\gamma \psi_g(\alpha+\varphi)\sigma(\alpha+\varphi)E_g/\psi_g(\varphi_0)\exp(-ik\rho\cos\alpha)\,\mathrm{d}\alpha\,(1+O(\epsilon^2)) \\
H_z(\rho,\varphi) &= \frac{1}{2\pi i}\int_\gamma \psi_f(\alpha+\varphi)\sigma(\alpha+\varphi)\xi_1(\alpha+\varphi)\exp(-ik\rho\cos\alpha)\,\mathrm{d}\alpha\,(1+O(\epsilon^2)).
\end{aligned}
\tag{17}
$$

The integral representation for $E_z(\rho,\varphi)$ is standard (Maliuzhinets 1959, Bernard 1987). The formula for $H_z$ determines admixtured polarization that arises due to the interaction of $E_z$ polarized wave with anisotropic phases. We have to consider meromorphic continuation of the function $\xi_1(\alpha)$ on the strip $|\mathrm{Re}\alpha| < \pi + \Phi$ in the integrand (17). Meromorphic continuation of $\xi_1(\alpha) = (K_1\zeta_0)(\alpha)$ on the strip $|\mathrm{Re}\alpha| < \pi + \Phi$ can be performed, using functional equations (9) ($m = 1$), or by means of the theory of S-integrals (Tuzhilin 1973).

Let us compute the asymptotic representation of $H_z$ when $k\rho \gg 1$. We use the saddle point technique. Let $\Phi > \pi/2$ and $\pi - \Phi < \varphi_0 < \Phi$ that means illumination of only one face of the wedge by the incident wave (figure 1). When the deformation of integration contour $\gamma$ into the steepest descent paths $\gamma_1$ through the saddle points $\pm\pi$ (figure 2) is performed, several poles of the integrand (17) can be captured. These poles are the following $\alpha_n^\pm = \pm 2\Phi - \varphi_0 - \varphi$, $\alpha_\theta^\pm = -\varphi \pm (\pi + \Phi + \theta^\pm)$, $\alpha_\chi^\pm = -\varphi \pm (\pi + \Phi + \chi^\pm)$. Using residue theorem, we obtain

$$
H_z \asymp r_\theta^\pm + r_\chi^\pm + r_n^\pm + \int_{\gamma 1} f(\alpha+\Phi)\exp(-ik\rho\cos\alpha)\,\mathrm{d}\alpha \tag{18}
$$

where $r_{\theta,\chi}^\pm$, $r_n^\pm$ are the terms contributed by the poles of integrand. The terms $r_\theta^\pm = A_\theta^\pm \exp[ik\rho\cos(\Phi + \theta^\pm \mp \varphi)]$ (and also $r_\chi^\pm$) represent leaky waves with complex phase functions. For large $k\rho$ we neglect these waves. It can be shown that the pole $\alpha_n^+ = 2\Phi - \varphi_0 - \varphi$ goes through the saddle point $+\pi$ for certain values of $\varphi$. So, we have the situation of coalescence of the pole and a saddle point. Instead of non-uniform representation (18), using the well known procedure (Borovikov and Kinber 1978), one can obtain a uniform one by $\varphi$ asymptotic formula

$$
\begin{aligned}
H_z \asymp{}& \exp(-ik\rho\cos[2\Phi - \varphi_0 - \varphi])(-\psi_f(2\Phi - \varphi_0)\xi_1(2\Phi - \varphi_0)) \\
&\times F\left[\sqrt{2k\rho}\cos\left[\frac{2\Phi - \varphi - \varphi_0}{2}\right]\right] + \frac{e^{ik\rho + i\pi/4}}{\sqrt{2\pi k\rho}}\left\{-\frac{\psi_f(2\Phi - \varphi_0)\xi_1(2\Phi - \varphi_0)}{2\cos((2\Phi - \varphi - \varphi_0)/2)}\right. \\
&\left. - \frac{\pi/(2\Phi)\cos(\pi\varphi_0/2\Phi)\psi_f(\pi + \varphi)\xi_1(\pi + \varphi)}{2\sin[(\pi/4\Phi)(\pi + \varphi - \varphi_0)]\cos[(\pi/4\Phi)(\pi + \varphi + \varphi_0)]}\right\} + \frac{e^{ik\rho + i\pi/4}}{\sqrt{2\pi k\rho}} \\
&\times \frac{\pi/(2\Phi)\cos(\pi\varphi_0/2\Phi)\psi_f(\varphi - \pi)\xi_1(\varphi - \pi)}{2\sin[(\pi/4\Phi)(\varphi - \pi - \varphi_0)]\cos[(\pi/4\Phi)(\varphi - \pi + \varphi_0)]}
\end{aligned}
\tag{19}
$$

where $F(x)$ is Frenel integral

$$
F(x) = \frac{1}{\sqrt{\pi i}}\int_{-\infty}^{x} e^{it^2}\,\mathrm{d}t.
$$

The value $\xi_1(2\Phi - \varphi_0)$ is computed explicitly, using functional equations (9) ($m = 1$) and the condition $\xi_1(\varphi_0) = 0$. The computations of $\xi_1(\varphi + \pi)$ exploit the mentioned meromorphic continuation of $\xi_1(\alpha)$ on the strip $|\mathrm{Re}\,\alpha| \leqslant \pi + \Phi$, but for $|\mathrm{Re}\,\alpha| < \Phi$ we have $\xi_1(\alpha) = (K_1\zeta_0)(\alpha)$ that is a rapidly converging integral. Frenel integral determines the wave field behaviour in the transition region in the vicinity of the ray corresponding to $2\Phi - \varphi = \pi + \varphi$. The second term in (19) has singularities due to the zeros of denominators, but the singularities cancel each other. Outside the transition region Frenel integral can be changed by asymptotics, and the wave field can be presented as a sum of an admixtured reflected wave and a cylindrical wave, spreading from the edge when $\varphi > 2\Phi - \varphi_0 - \pi$, and by a cylindrical wave only when $\varphi < 2\Phi - \varphi_0 - \pi$.

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